Q_4 Factorization of λK_n and λK_{m^x}

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Abstract

In this study, we show that necessary conditions for Q_4 factorization of λK_n and λK_{m^x} are also sufficient.

1 Introduction

Given a graph H, an H-decomposition of a graph G is a decomposition of the edge set of G into subgraphs isomorphic to H. Each subgraph H is called a block. Such a decomposition is called resolvable if it is possible to partition the blocks into classes (often referred to as parallel classes) such that every vertex of G appears in exactly one block of each parallel class.

A resolvable H-decomposition of G is generally referred to as an H-factorization of G, and each parallel class is called an H-factor of G. The case where $H = K_2$ (a single edge) is known as a 1-factorization since each vertex in each factor has degree 1. In general, if the factors are regular of degree k, then the factorization is called a k-factorization.

A k-partite (multipartite) graph is a graph whose vertices are partitioned into k different independent sets. When $k = 2$ these are bipartite graphs, and when $k = 3$, they are called tripartite graphs. If the parts have the same number of vertices, then the graph is called equipartite.

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The k-dimensional cube or k-cube is the graph with vertex set consisting of all binary vectors of length k and with edges joining pairs of vertices that differ in precisely one coordinate. This graph is k-regular and bipartite. The k-cube is denoted by Q_k. The total number of vertices in a k-cube is 2^k and the total number of edges is $\frac{2^k \cdot k}{2}$. In particular, Q_4, shown in Figure 1 has 16 vertices and 32 edges. The graph is regular of degree 4.

In 1979, two problems related to a k-cube decomposition and a k-cube factorization of complete graphs posed by Kotzig, Problems 15 and 16 in [8]. The two main open problems are:

Cube Decomposition Problem : For which values of n and k does there exist a k-cube decomposition of K_n?

Cube Factorization Problem : For which values of n and k does there exist a k-cube factorization of K_n?

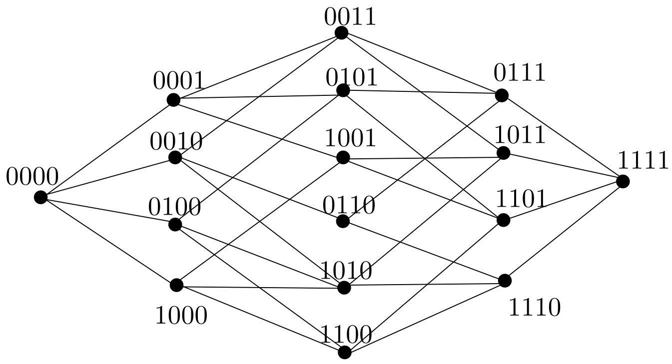

Figure 1: Q_4

Kotzig [8] established the necessary conditions for (K_n, Q_k)-design:

If there exists a (K_n, Q_k)-design; then either

(1) $n \equiv 1$ $(mod\ k2^k)$ or

(2) k is odd, $n \equiv 0$ $(mod\ 2^k)$ and $n \equiv 1$ $(mod\ k)$.

For even k, Kotzig [9] proved the sufficiency of the necessary conditions. Moreover, for $k = 3$ [10] and $k = 5$ [3], the problems have been solved completely. In addition, decomposition of λK_n into Q_3 is solved [1]. For each odd d, it is proved that there is an infinite arithmetic progression of even integers n for which a decomposition exists [7].

Since these problems were introduced, progress on the cube factorization problem has

been restricted to some special values of n, see [5] and [6]. The k-cube has 2^k vertices and is k-regular, so necessary conditions for the existence of a k-cube factorization of the complete graph K_n are

$$n \equiv 0 \ (mod \ 2^k) \text{ and } n \equiv 1 \ (mod \ k).$$

The first condition implies that n must be even and the second condition implies that n must have opposite parity to k. Hence, if a k-cube factorization of K_n exists, then k must be odd. For $k = 3$ [2] this problem is completely solved; the other cases are open yet.

If we consider λ-fold of complete graphs, then the necessary conditions will be

$$n \equiv 0 \ (mod \ 2^k) \text{ and } \lambda(n-1) \equiv 0 \ (mod \ k).$$

If the multipartite graph factorization is considered, the necessary conditions can easily be obtained as done for the complete graph case. Since a k-cube is a regular graph, in order to have a cube factorization, the graph should be regular. Thus, a multipartite graph should necessarily be equipartite. The necessary conditions for a λ-fold complete equipartite graph λK_{m^x}, where x is the number of parts and m is the number of independent vertices in each part, to have a Q_k-factorization are;

$$m \cdot x \equiv 0 \ (mod \ 2^k) \text{ and } \lambda m(x-1) \equiv 0 \ (mod \ k).$$

Wang [12] completed the case $k = 3$. The other cases are still open for $k > 3$.

In this study, we investigate the sufficiency of the necessary conditions for Q_4-factorization of λ-fold complete graphs and λ-fold multipartite graphs. Theorem 1.1 establishes the sufficiency of the necessary conditions.

Theorem 1.1. *The necessary conditions for λK_{m^x} and λK_n to have a Q_4-factorization are also sufficient.*

In order to prove Theorem 1.1, in section 2, some small examples and preliminary results about multipartite case are established. The proof of the main theorem is done in section 3. The result for the complete graph case is Corollary 3.4 which is obtained by using a subcase for complete multipartite graphs.

The following notions and results are helpful and used several times throughout this paper.

A complete equipartite graph has a 1-factorization if and only if the order is even [4].

Let k be a positive integer. A group divisible design (GDD), denoted by $k-GDD(m,n)$ is a triple (X,G,B) where:

1. X is a finite set of points of size $m \cdot n$,
2. G is a set of n subsets of X each with size m, called groups, which partition X,
3. B is a collection of subsets of X with size k, called blocks, such that every pair of points from distinct groups occurs in exactly one block, and
4. no pair of points belonging to a group occurs in any block.

A GDD is said to be resolvable and denoted by $k - RGDD(m,n)$ if its blocks can be partitioned into parallel classes each of which partitions the set of points. The following is a result of Sun and Ge [11]:

Theorem 1.2. *A $4 - RGDD(m,4)$ exists for every $m \in \mathbb{Z}^+$ except for $m \in \{2,3,6\}$.*

2 Preliminary Results

In this section, some important constructions and examples are given. These examples are used in section 3 for the proof of Theorem 1.1.

Example 2.1. *K_{4^4} has a factorization into Q_4's.*

Proof. Let the parts are denoted by A, B, C and D. $V(A) = \bigcup_{i=1}^{4} a_i$, $V(B) = \bigcup_{i=1}^{4} b_i$, $V(C) = \bigcup_{i=1}^{4} c_i$, $V(D) = \bigcup_{i=1}^{4} d_i$. Consider the following enumeration which gives the desired factorization:

	0000	0001	0010	0100	1000	0011	0101	1001	0110	1010	1100	0111	1011	1101	1110	1111
Q_1	a_1	c_1	c_2	c_3	c_4	a_2	a_3	a_4	b_1	b_2	b_3	d_1	d_2	d_3	d_4	b_4
Q_2	a_1	d_1	d_2	d_3	d_4	c_2	c_3	a_4	c_4	a_3	a_2	b_4	b_3	b_2	b_1	c_1
Q_3	a_1	b_1	b_2	b_3	b_4	d_3	d_2	a_4	d_1	a_3	a_2	c_1	c_2	c_3	c_4	d_4

□

Example 2.2. *$K_{8,8}$ has a factorization into Q_4's.*

Proof. Let A and B be the parts; $V(A) = \bigcup_{i=1}^{8} a_i$ and $V(B) = \bigcup_{i=1}^{8} b_i$. The following enumeration gives a Q_4-factorization of $K_{8,8}$:

	0000	0001	0010	0100	1000	0011	0101	1001	0110	1010	1100	0111	1011	1101	1110	1111
Q_1	a_1	b_1	b_2	b_3	b_4	a_2	a_3	a_4	a_5	a_6	a_7	b_5	b_6	b_7	b_8	a_8
Q_2	a_1	b_5	b_6	b_7	b_8	a_7	a_6	a_4	a_5	a_3	a_2	b_1	b_2	b_3	b_4	a_8

□

Example 2.3. *There exists a Q_4 factorization of K_{12^4}*

Proof. The parts of K_{12^4} are denoted by $A,\ B,\ C,\ D$ where each of these parts are divided into 3 sets denoted by A_i, B_i, C_i and D_i for $1 \le i \le 3$ containing 4 vertices each. Let $a_{i,j}$, $b_{i,j}$, $c_{i,j}$, $d_{i,j}$ denote the vertices of A_i, B_i, C_i, D_i respectively for $1 \le i \le 3$ and $1 \le j \le 4$.

The parts A_i, B_i, C_i and D_i form a K_{4^4} for each i which has 3 Q_4 factors by Example 2.1. Let these factors be denoted by $Q_{i,1}$, $Q_{i,2}$ and $Q_{i,3}$.

Consider Q_4's obtained by the following enumerations:

	0000	0001	0010	0100	1000	0011	0101	1001	0110	1010	1100	0111	1011	1101	1110	1111
$Q'_{1,1}$	$a_{2,1}$	$b_{3,1}$	$b_{3,2}$	$b_{3,3}$	$b_{3,4}$	$a_{2,2}$	$a_{2,3}$	$a_{2,4}$	$c_{2,1}$	$c_{2,2}$	$c_{2,3}$	$d_{3,1}$	$d_{3,2}$	$d_{3,3}$	$d_{3,4}$	$c_{2,4}$
$Q''_{1,1}$	$a_{3,1}$	$b_{2,1}$	$b_{2,2}$	$b_{2,3}$	$b_{2,4}$	$a_{3,2}$	$a_{3,3}$	$a_{3,4}$	$c_{3,1}$	$c_{3,2}$	$c_{3,3}$	$d_{2,1}$	$d_{2,2}$	$d_{2,3}$	$d_{2,4}$	$c_{3,4}$
$Q'_{1,2}$	$a_{2,1}$	$d_{3,4}$	$d_{3,3}$	$d_{3,2}$	$d_{3,1}$	$a_{2,2}$	$a_{2,3}$	$a_{2,4}$	$b_{2,4}$	$b_{2,3}$	$b_{2,2}$	$c_{3,1}$	$c_{3,2}$	$c_{3,3}$	$c_{3,4}$	$b_{2,1}$
$Q''_{1,2}$	$a_{3,1}$	$d_{2,4}$	$d_{2,3}$	$d_{2,2}$	$d_{2,1}$	$a_{3,2}$	$a_{3,3}$	$a_{3,4}$	$b_{3,4}$	$b_{3,3}$	$b_{3,2}$	$c_{2,1}$	$c_{2,2}$	$c_{2,3}$	$c_{2,4}$	$b_{3,1}$
$Q'_{1,3}$	$a_{2,1}$	$c_{3,4}$	$c_{3,3}$	$c_{3,2}$	$c_{3,1}$	$a_{2,2}$	$a_{2,3}$	$a_{2,4}$	$d_{2,1}$	$d_{2,2}$	$d_{2,3}$	$b_{3,4}$	$b_{3,3}$	$b_{3,2}$	$b_{3,1}$	$d_{2,4}$
$Q''_{1,3}$	$a_{3,1}$	$c_{2,4}$	$c_{2,3}$	$c_{2,2}$	$c_{2,1}$	$a_{3,2}$	$a_{3,3}$	$a_{3,4}$	$d_{3,1}$	$d_{3,2}$	$d_{3,3}$	$b_{2,4}$	$b_{2,3}$	$b_{2,2}$	$b_{2,1}$	$d_{3,4}$

Apply the permutation to obtain new Q_4's: $(a_{2,j}, a_{1,j})(b_{2,j}, b_{1,j})(c_{2,j}c_{1,j})(d_{2,j}, d_{1,j})$.

Rename new Q_4's by the transformation: $Q'_{1,j} \to Q'_{2,j}, \qquad Q''_{1,j} \to Q''_{2,j}$.

Independently, apply the permutation to obtain new Q_4's: $(a_{3,j}, a_{1,j})(b_{3,j}, b_{1,j})(c_{3,j}c_{1,j})(d_{3,j}, d_{1,j})$.

Rename new Q_4's by the transformation: $Q'_{1,j} \to Q'_{3,j}, \qquad Q''_{1,j} \to Q''_{3,j}$.

Then, $Q_{i,j} \cup Q'_{i,j} \cup Q''_{i,j}$ for $1 \le i \le 3$ and $1 \le j \le 3$ form the factors of the factorization of K_{12^4}. □

Example 2.4. *There exists a Q_4 factorization of $2K_{2^8}$.*

Proof. Let the parts are denoted by A, B, C, D, E, F, G, H where each part has 2 vertices. Consider the following 4 factors:

	0000	0001	0010	0100	1000	0011	0101	1001	0110	1010	1100	0111	1011	1101	1110	1111
Q_1	a_1	b_1	b_2	d_2	h_2	a_2	c_2	g_2	c_1	g_1	e_1	d_1	h_1	f_1	f_2	e_2
Q_2	a_1	e_1	c_1	f_2	g_2	g_1	b_2	c_2	h_2	e_2	d_1	d_2	a_2	h_1	b_1	f_1
Q_3	a_1	d_1	c_2	h_1	e_2	b_2	e_1	h_2	f_2	g_1	d_2	g_2	f_1	a_2	b_1	c_1
Q_4	a_1	b_1	b_2	g_1	f_1	a_2	h_1	e_1	h_2	e_2	d_1	g_2	f_2	c_1	c_2	d_2

The remaining 3 factors are obtained by considering $A \cup B$, $C \cup D$, $E \cup F$, $G \cup H$ as the parts of K_{4^4} and taking the factors as done in Example 2.1. □

Lemma 2.5. *There exists a Q_4-factorization of $K_{(16k+4)^4}$ for $k \geq 0$.*

Proof. The number of parallel classes will be $12k+3$ and the number of Q_4's in each parallel class will be $4k+1$.

There exists a $4-RGDD(4k+1,4)$ for $k \geq 0$ by Theorem 1.2. Let $b_{i,j}$ be the j^{th} block of the i^{th} parallel class. For each $1 \leq i \leq 4k+1$ and $1 \leq j \leq 4k+1$ blow up vertices in each block by 4 to obtain a copy of K_{4^4} on $b_{i,j} \times \{1,2,3,4\}$ for each i and j. Then place a Q_4-factorization of K_{4^4} on the blown up blocks. There are 3 factors in each Q_4-factorization of K_{4^4} by Example 2.1; let the factors for each blown up b_{ij} be $Q_{i,j,k}$ for $1 \leq k \leq 3$. Then the following are the factors of Q_4-factorization:

$$\pi_{i,1} = \bigcup_{j=1}^{4k+1} Q_{i,j,1}, \quad \pi_{i,2} = \bigcup_{j=1}^{4k+1} Q_{i,j,2}, \quad \pi_{i,3} = \bigcup_{j=1}^{4k+1} Q_{i,j,3},$$

where $1 \leq i \leq 4k+1$. The number of parallel classes is $12k+3$ and the number of Q_4's in each parallel class is $4k+1$ as expected. □

Lemma 2.6. *There exists a Q_4-factorization of $K_{(16k+12)^4}$ for $k \geq 0$.*

Proof. Since there exists $4-RGDD(4k+3,4)$ for $k \geq 1$ by Theorem 1.2, a Q_4-factorization of $K_{(16k+12)^4}$ for $k \geq 1$ can be obtained as in the proof of Lemma 2.5. The case $k=0$ is Example 2.3. □

Lemma 2.7. *There exists a Q_4-factorization of $K_{(16k)^t}$ and $K_{(8k)^{2t}}$ for $k,t \geq 1$.*

Proof. Consider a 1-factorization of $K_{(2k)^t}$ which is known to exist since the number of vertices is even. Let the factors be $F_1, F_2, ..., F_n$ where $n = (2k)(t-1)$. Let the edges of the factors F_i be $E(F_i) = \{e_{i,1}, e_{i,2}, ..., e_{i,s}\}$ where $s = k \cdot t$. When each vertex of the original graph is blown up by 8, then each edge in the 1-factors correspond to a $K_{8,8}$. By Example 2.2, $K_{8,8}$ has a Q_4-factorization into 2 Q_4's. Let $Q_{i,j,1}$, $Q_{i,j,2}$ be the Q_4 factors of each $K_{8,8}$ corresponding to the edge $e_{i,j}$. Hence, the parallel classes of this factorization are:

$$\pi_{i,1} = \bigcup_{j=1}^{s} Q_{i,j,1}, \quad \pi_{i,2} = \bigcup_{j=1}^{s} Q_{i,j,2} \quad \text{for } 1 \leq i \leq 2k(t-1).$$

Similarly, consider 1-factorization of $K_{(k)^{2t}}$. Let the factors be $F_1, F_2, ..., F_n$ where $n = (k)(2t-1)$. Let the edges of the factors F_i be $E(F_i) = \{e_{i,1}, e_{i,2}, ..., e_{i,s}\}$ where $s = k \cdot t$.

When each vertex of the original graph is blown up by 8, then each edge in the 1-factors correspond to a $K_{8,8}$. Let $Q_{i,j,1}$, $Q_{i,j,2}$ be the Q_4 factors of each $K_{8,8}$ corresponding to the edge $e_{i,j}$. Hence, the parallel classes of this factorization are:

$$\pi_{i,1} = \bigcup_{j=1}^{s} Q_{i,j,1}, \quad \pi_{i,2} = \bigcup_{j=1}^{s} Q_{i,j,2} \quad \text{for } 1 \leq i \leq k(2t-1). \qquad \square$$

3 4-Cube Factorization of Complete Multipartite and Complete Graphs

There are several cases to be considered for the factorization of λ-fold complete multipartite graphs λK_{m^x}. The cases depend generally on the value of λ and there will be subcases where different values of the other parameters are considered. This problem will be solved by casing on m rather than λ.

Case 1 : $\lambda \equiv$ 1 or 3 (mod 4)

The necessary conditions reduce to

$$m \cdot x \equiv 0 \ (mod\ 16) \text{ and } m \equiv 0 \ (mod\ 4).$$

4 subcases will be considered depending on the value of m,.

1.1 m $\equiv$ 4, 12 (mod 16)

For the first condition to be satisfied, x should be a multiple of 4. So, we are looking for a Q_4-factorization of $K_{(16k+4)^{4t}}$ and $K_{(16k+12)^{4t}}$ for $k \geq 0$, $t \geq 1$.

Consider the vertex disjoint subgraphs H_i of $K_{(16k+4)^{4t}}$ and $K_{(16k+12)^{4t}}$ where each H_i is isomorphic to $K_{(16k+4)^4}$ and $K_{(16k+12)^4}$ for $1 \leq i \leq t$. By Lemmas 2.5 and 2.6, each H_i has a Q_4 factorization.

The remaining edges correspond to $K_{(64k+16)^t}$ and $K_{(64k+48)^t}$ respectively. By Lemma 2.7, these graphs have Q_4 factorization.

Combining these factors gives the factorization of $K_{(16k+4)^{4t}}$ and $K_{(16k+12)^{4t}}$ respectively.

1.2 m $\equiv$ 0, 8 (mod 16)

Both of the necessary conditions are satisfied when $m \equiv 0 \ (mod\ 16)$. So, we are looking for factorization of $K_{(16k)^t}$ which is immediate from Lemma 2.7.

For $m \equiv 8$ $(mod\ 16)$, to satisfy the first necessary condition, x should be a multiple of 2. So, we are looking for a Q_4-factorization of $K_{(16k+8)^{2t}}$ which is again immediate from Lemma 2.7.

Case 2 : $\lambda \equiv \mathbf{2}$ **(mod 4)**

For this case, the necessary conditions reduce to $m \equiv 0$ $(mod\ 2)$ and $m \cdot x \equiv 0$ $(mod\ 16)$. When $m \equiv 0$ $(mod\ 4)$, this problem is solved in Case 1. So, we are looking for $m \equiv 2$ $(mod\ 4)$ case. Lemma 3.3 establishes the general version of this. Lemma 3.1 and 3.2 help us to prove Lemma 3.3.

Lemma 3.1. *There exists a Q_4 factorization of $2K_{16^2} - 2F$ where $2F$ represents 2 copies of a 2-factor with 4-cycles.*

Proof. Let the parts are denoted as A and B and the vertices are a_i and b_i respectively for $1 \le i \le 16$. Let F denote the following 4-cycle: $F = \bigcup_{i=1}^{8}(a_{2i-1}, b_{2i}, a_{2i}, b_{2i-1})$.

Consider the following factors:

	0000	0001	0010	0100	1000	0011	0101	1001	0110	1010	1100	0111	1011	1101	1110	1111
Q_1	a_1	b_3	b_4	b_5	b_7	a_2	a_7	a_5	a_8	a_6	a_3	b_6	b_8	b_1	b_2	a_4
Q'_1	a_9	b_{11}	b_{12}	b_{13}	b_{15}	a_{10}	a_{15}	a_{13}	a_{16}	a_{14}	a_{11}	b_{14}	b_{16}	b_9	b_{10}	a_{12}
Q_2	a_3	b_6	b_5	b_8	b_2	a_4	a_1	a_7	a_2	a_8	a_5	b_7	b_1	b_4	b_3	a_6
Q'_2	a_{11}	b_{14}	b_{13}	b_{16}	b_{10}	a_{12}	a_9	a_{15}	a_{10}	a_{16}	a_{13}	b_{15}	b_9	b_{12}	b_{11}	a_{14}
Q_3	a_1	b_3	b_5	b_6	b_8	a_7	a_8	a_6	a_2	a_4	a_3	b_4	b_2	b_1	b_7	a_5
Q'_3	a_9	b_{11}	b_{13}	b_{14}	b_{16}	a_{15}	a_{16}	a_{14}	a_{10}	a_{12}	a_{11}	b_{12}	b_{10}	b_9	b_{15}	a_{13}

$Q_i \cup Q'_i$ $1 \le k \le 3$ gives the 3 factors of the factorization.

Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ where $V(A_1) = \bigcup_{i=1}^{8} a_i$, $V(A_2) = \bigcup_{i=9}^{16} a_i$, $V(B_1) = \bigcup_{i=1}^{8} b_i$, $V(B_2) = \bigcup_{i=9}^{16} b_i$.

The edges between A_1 and B_2; B_1 and A_2 construct $2K_{8,8}$ which has Q_4 factorization by Example 2.2. The remaining 4 factors are obtained from this factorization. $\square$

Lemma 3.2. *There exists a Q_4 factorization of $2K_{(2k)^8}$ for $k \ge 1$.*

Proof. Example 2.4 is the construction for $k = 1$. If k is even, Case 1 gives the result. Let k be odd and $k \ge 3$.

There exists a near 1-factorization of K_k for odd k [4]. For each near 1-factor, let the

vertex denote $2K_{2^8}$ and each edge denote $2K_{16^2} - 2F$ where F represents 2-factor shown in Lemma 3.1. The factors of those graphs were constructed in Example 2.4 and Lemma 3.1. The desired factorization is obtained by considering these factors for each near 1-factor of K_k. □

Lemma 3.3. *There exists a Q_4 factorization of $2K_{(2k)^{8t}}$ for $k \geq 1$ and $t \geq 1$.*

Proof. Let H_i $1 \leq i \leq t$ denote the vertex disjoint subgraphs of $2K_{(2k)^{8t}}$ where each H_i is isomorphic to $2K_{(2k)^8}$. By Lemma 3.2, H_i has a factorization.

Consider the remaining edges which construct $2K_{(16k)^t}$. This graph has a factorization by Lemma 2.7.

Combining these factors gives the factorization of $2K_{(2k)^{8t}}$. □

Case 3 : $\lambda \equiv \mathbf{0}$ **(mod 4)**

For this situation, the necessary conditions reduce to $m \cdot x \equiv 0$ $(mod\ 16)$. So there are 5 subcases to be considered.

3.1 x $\equiv \mathbf{0}$ **(mod 16)**

For this case, m is arbitrary; so, we are looking for factorization of $4K_{k^{16t}}$.

Consider the vertex disjoint subgraphs H_i of $4K_{k^{16t}}$ where each H_i is isomorphic to $4K_{k^{16}}$ for $1 \leq i \leq t$.

To get a factorization of H_i, consider a resolvable design of order 16 with block size 4. Let each parallel class is denoted by K_{4^4}. If each vertex is blown up by k, then the new parallel classes correspond to a $K_{(4k)^4}$. Notice that the edge-disjoint union of new built parallel classes gives $4K_{k^{16t}}$. $K_{(4k)^4}$ has a Q_4 factorization by Case 1 and the number of factors is $3k$. Let $\pi_{i,j,l}$ denote the Q_4 factors of H_i for j^{th} parallel class of $K_{(4k)^4}$; $1 \leq i \leq t$, $1 \leq j \leq 5$, $1 \leq l \leq 3k$. Then the followings are the factors of the factorization:

$$\bigcup_{i=1}^{t} \pi_{i,j,l} \text{ for } 1 \leq j \leq 5 \text{ and } 1 \leq l \leq 3k.$$

The remaining edges correspond to a $4K_{(16k)^t}$. By Lemma 2.7, this graph has a Q_4 factorization.

Corollary 3.4. *A λ-fold complete graph satisfying the necessary conditions has a Q_4 factorization.*

Proof. The necessary conditions for Q_4 factorization of λK_n are:

$$n \equiv 0 \ (mod\ 16) \text{ and } \lambda(n-1) \equiv 0 \ (mod\ 4).$$

Since n is divisible by 4, the necessary conditions reduce to

$$n \equiv 0 \ (mod\ 16) \text{ and } \lambda \equiv 0 \ (mod\ 4).$$

So, the factorization of $4\lambda' K_{16t}$ is required. This is immediate from the previous subcase by taking $k = 1$. $\square$ and λ' copies of the factors.

3.2 x $\equiv$ 8 (mod 16)

For this case, $m \equiv 0$ $(mod\ 2)$; so, we are looking for a factorization of $4K_{(2k)^{8t}}$. Lemma 3.3 constructs a factorization of $2K_{(2k)^{8t}}$. 2 copies of the factors give the desired factorization.

The remaining subcases $4K_{(4k)^{4t}}$, $4K_{(8k)^{2t}}$ and $4K_{(16k)^{t}}$ have Q_4-factorizations by Case 1. So the case where $\lambda \equiv 0$ $(mod\ 4)$ is completed.

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